NONSTEADY STREAMLINING OF A CYLINDER BY AN INCOMPRESSIBLE CONDUCTIVE FLUID

V. I. Tokatly

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Let us consider the following problem. An infinite circular cylinder composed of a dielectric is placed in an incompressible inviscid fluid flow. At infinity the flow is homogeneous and the velocity is perpendicular to the cylinder axis. A current-carrying conductor, creating a magnetic field, is arranged on the axis of the cylinder. At $t = 0$ the conductivity $\sigma = \text{const}$ is "switched on," i.e., the fluid flowing over the cylinder becomes conductive. It is assumed that the magnetic Reynolds number $N_{\text{Rem}} = V_0 r_0 / \nu_m \ll 1$ and that the induced magnetic field can be neglected; the electric field $\mathbf{E} = 0$ and the current is found from the relation $j = \sigma \mathbf{v} \times \mathbf{H}/c$.

Since in this case the velocity and magnetic field vectors lie in the same plane, the equation of motion [1] reduces, after application of the curl, to the form:

$$\frac{\partial}{\partial\tau}(\operatorname{rot}\mathbf{u}) + (\mathbf{u}\cdot\nabla)\operatorname{rot}\mathbf{u} = \beta(\mathbf{h}\cdot\nabla)\mathbf{u}\times\mathbf{h}, \tag{1}$$

$$R = r_0 r, \quad \mathbf{H} = H_0 \mathbf{h}, \quad \mathbf{v} = V_0 \mathbf{u}, \quad t = T_0 \tau, \quad T_0 = \frac{r_0}{V_0},$$

$$\beta = \frac{r_0 V_0 H_0^2}{4\pi\rho_0 V_0^2 \nu_m}, \qquad \nu_m = \frac{c^2}{4\pi\sigma}.$$

Here, H_0 is the magnetic field at the edge of the cylinder, V_0 is the velocity of the homogeneous flow at infinity, r_0 is the cylinder radius, ρ_0 the fluid density, $\mathbf{v}$ is the velocity, $\mathbf{H}$ is the magnetic field strength, and c is the speed of light. We find the solution of (1) in the form:

$$\mathbf{u} = \mathbf{u}_0 + \tau\mathbf{u}_1 + \tau^2\mathbf{u}_2 + \ldots. \tag{2}$$

Substituting (2) into (1), we obtain the following system for determining $\mathbf{u}_n$:

$$\operatorname{rot}\mathbf{u}_0 = 0, \qquad \operatorname{rot}\mathbf{u}_1 = \beta(\mathbf{h}\cdot\nabla)\mathbf{u}_0\times\mathbf{h}, \ldots, \tag{3}$$

$$\operatorname{rot}\mathbf{u}_n = \frac{\beta}{n}(\mathbf{h}\cdot\nabla)\mathbf{u}_{n-1}\times\mathbf{h} - \frac{1}{n}\sum_{k=0}^{n-2}(\mathbf{u}_k\cdot\nabla)\operatorname{rot}\mathbf{u}_{n-(k+1)}.$$

Using the continuity equation

$$\operatorname{div}\ \mathbf{u} = 0 \tag{4}$$

we can find $\mathbf{u}_n$ in terms of the stream function Ψ_n,

$$u_{nr} = \partial\psi_n / r\partial\theta, \qquad u_{n\theta} = -\partial\psi_n/\partial r.$$

The boundary conditions are:

$$\psi_{n/r=1} = 0, \qquad \psi_n \to 0 \quad \text{as} \quad r\to\infty \qquad (n \geqslant 1), \qquad \psi_0 = (r - r^{-1})\sin\theta.$$

As $\tau \to 0$, from (3) we easily find

$$\psi = (r - r^{-1})\sin\theta + {}^1/_2\,\beta\tau\,[{}^1/_4\,(r^{-1} - r^{-3}) - r^{-1}\ln r]\sin\theta + \tau^2\{{}^1/_2(\beta/4)^2\,[{}^7/_{12}r^{-1} - {}^1/_2\,r^{-3} - {}^1/_{12}\,r^{-5} - r^{-3}\ln r]\sin\theta + \beta\,[{}^{13}/_{96}\,r^{-2} - {}^1/_6\,r^{-4} + {}^1/_{32}\,r^{-6} - {}^1/_4\,r^{-2}\ln r]\sin 2\theta\} + O(\tau^3).$$

As $r \to 1$, setting $r = 1 + \rho(\rho \ll 1)$, we have

$$\psi = \rho \, [(2 - {}^1\!/_4\, \beta\tau + {}^1\!/_{96}\, (\beta\tau)^2) \sin\theta - {}^1\!/_{24}\, \beta\tau^2 \sin 2\theta] \, ,$$
$$u_r = \rho \, [(2 - {}^1\!/_4\, \beta\tau + {}^1\!/_{96}\, (\beta\tau)^2) \cos\theta - {}^1\!/_{12}\, \beta\tau^2 \cos 2\theta] \, .$$

From the last expression it follows that the streamlines "move away" from the cylinder and the fluid is decelerated,. the change in velocity being greater near the rear than near the forward stagnation point.

We will now consider the flow as $\beta \to 0$. In this case the velocity field is only slightly disturbed, and we can rewrite (1),

$$\frac{\partial}{\partial\tau} \operatorname{rot} \mathbf{u} + (\mathbf{u}_0 \cdot \nabla) \operatorname{rot} \mathbf{u} = \beta \, (\mathbf{h} \cdot \nabla) \, \mathbf{u}_0 \times \mathbf{h} \, , \tag{5}$$

while (3) goes over into the system

$$\Delta\psi_0 = 0, \quad \Delta\psi_1 = -\beta \, (\mathbf{h} \cdot \nabla)^2 \psi_0, \; . \; . \; . \; , \quad \Delta\psi_n = -\frac{1}{n} (\mathbf{u}_0 \cdot \nabla) \, \Delta\psi_{n-1} \quad (n \geqslant 2) \, . \tag{6}$$

The solution of (6) can be represented in the form:

$$\psi_0 = (r - r^{-1}) \sin\theta, \qquad \psi_n = \beta \sum_{k=0}^{q} R_{nk} \sin (n - 2k) \, \theta \quad (n \geqslant 1) \, , \tag{7}$$

$$R_{nk} = c_{nk} r^{2k-n} + \sum_{m=0}^{n} b_{nkm} \left[(1 - \delta_{0k}\delta_{nm}) \, r^{2m-3n} + \delta_{0k}\delta_{nm} \frac{\ln r}{r^n} \right] ,$$

$$c_{nk} = -\sum_{m=0}^{n} b_{nkm} \, (1 - \delta_{0k}\delta_{nm}), \qquad b_{nkm} = \frac{a_{nkm}}{(2m - 3n)^2 - (n - 2k)^2} \begin{pmatrix} m \neq n \\ k \neq 0 \end{pmatrix} b_{n0n} = -\frac{a_{n0n}}{2n}$$

$$a_{nkm} = \frac{1}{n} \left\{ (n + k - m) \, [(1 - \delta_{0k}) \, (1 - \delta_{0m}) \, \frac{1 - \delta_{qk} \, (-1)^n}{1 + \delta_{qk}} \, a_{n-1,\, k-1,\, m-1} - \right.$$
$$- (1 - \delta_{nm}) \, \frac{1 + \delta_{qk} \, (-1)^n}{1 + \delta_{qk}} \, a_{n-1,\, k,\, m}] + (2n - k - m) \left[(1 - \delta_{0m}) \, \frac{1 + \delta_{qk} \, (-1)^n}{1 + \delta_{qk}} \times \right.$$
$$\left. \left. \times \, a_{n-1,\, k,\, m-1} - (1 - \delta_{0k}) \, (1 - \delta_{nm}) \, \frac{1 - \delta_{qk} \, (-1)^n}{1 + \delta_{qk}} \, a_{n-1,\, k-1,\, m} \right] \right\}$$

$$q = q_n = q_{n-1} + {}^1\!/_2 \, [1 - (-1)^n], \qquad q_1 = 0, \qquad a_{100} = -1, \qquad a_{101} = 1 \, .$$

Here, δ_{mm} is the Kronecker delta. From an estimate of the discarded terms it follows that the obtained solution (7) is applicable when $\tau \ll 1/\beta$.

Simpler expressions are obtained in the other limiting case in which $\beta \to \infty$. Introducing the new variable. $\tau_1 = \beta\tau$, we reduce (1) to the form

$$\beta \frac{\partial \operatorname{rot} \mathbf{u}}{\partial\tau_1} + (\mathbf{u} \cdot \nabla) \operatorname{rot} \mathbf{u} = \beta \, (\mathbf{h} \cdot \nabla) \, \mathbf{u} \times \mathbf{h} \, . \tag{8}$$

As $\beta \to \infty$, we neglect the second term in (8) and obtain the following equation for the stream function ψ:

$$\partial \, (\Delta\psi) \, / \, \partial\tau_1 = -(\mathbf{h} \cdot \nabla)^2 \, \psi \, . \tag{9}$$

The solution of (9) can be represented in the form:

$$\psi = \sin\theta \sum_{n=0} R_n = R \sin\theta, \qquad R_0 = r - r^{-1} \, , \tag{10}$$

$$R_n = \frac{1}{r^{2n+1}} \left\{ (A_{n0} r^2 \ln r - B_{n0}) + \sum_{k=1}^{n} r^{2k} \, (A_{nk} - B_{nk}) \right\} , \tag{11}$$

$$A_{n0} = \frac{\tau_1}{4n^2 \, (n - 1)} \, A_{n-1,0}, \quad A_{n1} = \frac{\tau_1}{4n^2 \, (n - 1)} \left[\frac{2n - 1}{2n \, (n - 1)} \, A_{n-1,0} + A_{n-1,1} \right] ,$$

$$A_{10} = -\frac{\tau_1}{2} \, ,$$

$$A_{11} = 0, \quad A_{nk} = \frac{\tau_1}{4n \, (n + 1 - k) \, (n - k)} \, A_{n-1,\, k} \quad (2 \leqslant k \leqslant n - 1), \quad A_{nn} = -\sum_{k=1}^{n-1} A_{nk} \, ,$$

$$B_{nk} = \frac{\tau_1}{4n \, (n + 1 - k) \, (n - k)} \, B_{n-1,\, k} \quad (0 \leqslant k \leqslant n - 1), \quad B_{nn} = -\sum_{k=0}^{n-1} B_{nk}, \quad B_{00} = 1 \, .$$

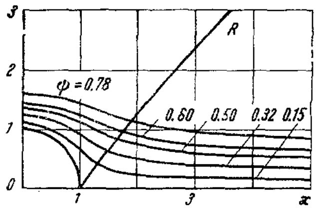

Fig. 1

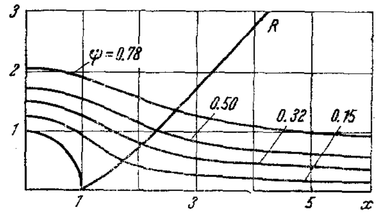

Fig. 2

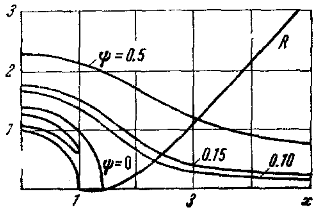

Fig. 3

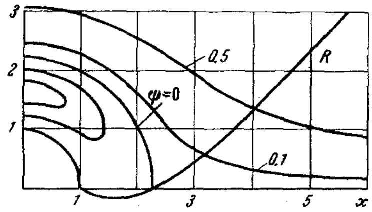

Fig. 4

Values of R were calculated on a computer for several τ_1. The graph of R and the streamlines are presented in Fig. 1 for $\tau_1 = 2$. It is clear that at $\tau_1 = 2$ the velocity field was still only slightly deformed. Figure 2 corresponds to $\tau_1 = 8$. The streamlines have moved further away from the cylinder, and R curve is flatter, which corresponds to a decrease in velocity near the cylinder. At $\tau_1 = 16$ a region develops near the cylinder in which the fluid circulates (Fig. 3). As τ_1 increases the region of trapped fluid increases (Fig. 4, $\tau_1 = 32$).

It should be noted that if a stationary solution exists, when $\beta \ll 1$ the velocity field in the stationary regime must differ significantly from that shown in Fig. 4, since, in the stationary regime, flow with closed streamlines is impossible. To demonstrate this, we write the equation of motion in the form [1]:

$$\rho_0 [\partial \mathbf{v}/\partial t + (\mathbf{v}\cdot\nabla)\mathbf{v}] = -\nabla p + \mathbf{j} \times \mathbf{H}/c . \tag{12}$$

Integrating (12) along the streamline, we obtain for $\partial v/\partial t = 0$

$$\oint v H_\perp^2 dl_v = 0 . \tag{13}$$

The last relation can be satisfied at $\nu \neq 0$ only when $H_\perp = 0$ i.e., when the streamline coincides with a line of magnetic force. In the problem considered this condition is not satisfied. The above-noted singularity of solution (10) is associated with the fact that it is applicable at $\tau_1 \ll \beta$.

In conclusion we note that if the current in the conductor varies with time, the magnetic field is determined from the equations

$$\Delta a = N_{Rem} (\partial a / \partial \tau + \mathbf{u}\cdot\nabla a) \quad \text{at} \quad r > 1, \qquad \Delta a = -i\delta(r) \quad \text{at} \quad r < 1 .$$

At $N_{Rem} \ll 1$ the right-hand side of the first equation can be replaced by zero, if $T_*/T_i \ll 1$ ($T_* = r_0^2/\nu_m$ is the characteristic time of variation of the current in the conductor). In this case the magnetic field (correct to N_{Rem}) is

$$h_\theta = i(\tau) / r, \qquad h_r = 0 .$$

It is easy to prove that if the current in the conductor varies according to a power law

$$i(\tau) = \tau^m$$

and, moreover, $\beta \ll 1$, then correct to terms $\sim 1/\beta$ the stream function is given by the expression

$$\psi = \sin\theta \sum_{n=0} R_n \left(\frac{2m+1}{\beta}\right)^{2mn} \left(\frac{\tau_1}{2m+1}\right)^{(2m+1)\,n} \tag{14}$$

where R_n is given by Eq. (11) and by setting $\tau_1 = 1$.

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REFERENCE

1. L. D. Landau and E. M. Lifshits, The Electrodynamics of Coutinuous Media [in Russian], Gostekhizdat, 1957.

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